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# COTYPE 2 ESTIMATES FOR SPACES OF POLYNOMIALS ON SEQUENCE SPACES

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ABSTRACT

We give asymptotically correct estimations for the cotype 2 constant $\mathbf{C}_2(\mathcal{P}(^m X_n))$ of the space $\mathcal{P}(^m X_n)$ of all $m$-homogeneous polynomials on $X_n$, the span of the first $n$ sequences $e_k = (\delta_{kj})_j$ in a Banach sequence space $X$. Applications to Minkowski, Orlicz and Lorentz sequence spaces are given.

## 1. Introduction

A scalar valued mapping $P$ on a Banach space $E$ is said to be a continuous $m$-homogeneous polynomial provided that there is some $m$-linear and continuous form $L$ on $E \times \cdots \times E$ ($m$ times) such that $L(x, \ldots, x) = P(x)$ for all $x \in E$. The space

$$\mathcal{P}(^m E) = \{P\colon P\ m\text{-homogeneous polynomial on } E\}$$

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together with the norm

$$\|P\| = \sup_{\|x\|\leq 1} |P(x)|$$

forms a Banach space. A (bounded linear) operator $T\colon E \to F$ between Banach spaces is said to have cotype 2 whenever there exists some finite constant $\kappa \geq 0$ such that for any finitely many $x_1, \ldots, x_n \in E$,

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^2\Big)^{1/2} \leq \kappa \Big(\int_0^1 \|\sum_{k=1}^{n} r_k(t) x_k\|^2 dt\Big)^{1/2},$$

where $r_k$ stands for the $k$-th Rademacher function. As usual, we denote by $\mathbf{C}_2(T)$ the best constant in this inequality. It is well known that the cotype 2 operators with $\mathbf{C}_2$ form a Banach operators ideal. A Banach space $E$ is said to have cotype 2 if $id_E$ has cotype 2; we denote $\mathbf{C}_2(E)$ for $\mathbf{C}_2(id_E)$.

Dineen proved in [9], Proposition 1.54 (see also [8]) that $\ell_\infty$ is finitely representable in $\mathcal{P}(^mE)$ (provided $m \geq 2$ and $E$ is infinite dimensional), which in particular means that $\mathcal{P}(^mE)$ does not have cotype 2.

Let now $X$ be a Banach sequence space (e.g., $\ell_p$, $1 \leq p \leq \infty$) and denote by $X_n$ its subspace spanned by the sequences $e_k = (\delta_{kj})_j$, $1 \leq k \leq n$. Since $\mathcal{P}(^mX)$ for $m \geq 2$ has no cotype 2, the sequence $(\mathbf{C}_2(\mathcal{P}(^mX_n)))_n$ tends to $\infty$. Our aim is to give an asymptotically optimal description of this divergence. We conjecture that for any symmetric real Banach sequence space $X$ and $m \geq 2$,

$$\mathbf{C}_2(\mathcal{P}(^mX_n)) \asymp (n^{1/2})^{m-1}\mathbf{C}_2(X_n'), \tag{1}$$

and prove this conjecture within the class of all $X$ which are either 2-convex or are 2-concave and have non-trivial convexity. This result gives various asymptotically optimal cotype 2 estimates for spaces of polynomials on $\ell_p$-spaces, Orlicz spaces and Lorentz spaces.

In an appendix we give an asymptotically correct estimate for the cotype 2 constants of the injective tensor product of finite dimensional spaces which is a proper extension of results from [1].

## 2. Preliminaries

We use standard notation for Banach spaces and sequence spaces. For background on Banach space theory we refer to [14], [15], for local theory of Banach spaces to [4], [7], [23] and for background on general theory of polynomials and tensor products see [4], [9], [10]. By $\mathcal{L}(E;F)$ we denote all (bounded and linear) operators between the Banach spaces $E$ and $F$.

Given any two sequences $(a_n)_n, (b_n)_n \subseteq \mathbb{R}$, we write $a_n \prec b_n$ whenever there is some $c > 0$ such that $a_n \leq c\, b_n$ for all $n$. We write $a_n \asymp b_n$ if $a_n \prec b_n$ and $b_n \prec a_n$.

Throughout the paper $X$ will denote a Köthe function space (in the sense of [15], Definition 1.b.17) over $(\mathcal{J}, \Sigma, \mu)$, where $\mathcal{J}$ is finite or countable and $\mu$ is the counting measure on $\mathcal{J}$, i.e., a real Banach space $X$ of functions $\phi: \mathcal{J} \to \mathbb{R}$ such that

(i) if $|\psi(j)| \leq |\phi(j)|$ for all $j$ and $\phi \in X$, then $\psi \in X$ and $\|\psi\| \leq \|\phi\|$,

(ii) for all finite $\mathcal{I} \subseteq \mathcal{J}$, the characteristic function $\chi_{\mathcal{I}}$ belongs to $X$.

Our main reason for dealing with real spaces only is to keep the paper simple; in section 7 we indicate how our main results on (1) can be transferred to the complex case.

We will say that $X$ is a Banach sequence space whenever $\mathcal{J} = \mathbb{N}$ and $\ell_1 \hookrightarrow X \hookrightarrow \ell_\infty$ with embeddings of norm 1. In particular, all $e_n = (\delta_{nk})_k \in X$ and $\|e_n\| = 1$. A Banach sequence space is said to be symmetric if every $\xi \in X$, when we consider its decreasing rearrangement $(\xi_n^\star)_{n \in \mathbb{N}}$ given by

$$\xi_n^\star := \inf\{\sup_{i \in \mathbb{N} \setminus J} |\xi_i| : J \subseteq \mathbb{N}, \operatorname{card}(J) < n\},$$

satisfies that $\|(\xi_n)_n\|_X = \|(\xi_n^\star)_n\|_X$. For each $n \in \mathbb{N}$ we define the space $X_n := \operatorname{span}\{e_1, \ldots, e_n\}$. Recall the definition of the Köthe dual of a Köthe function space $X$ modeled on a finite or countable set $\mathcal{J}$:

$$X^\times := \{\eta \in \mathbb{R}^{\mathcal{J}} : \eta\xi \in \ell_1(\mathcal{J}) \text{ for all } \xi \in X\}.$$

With the norm $\|\eta\|_{X^\times} := \sup_{\|\xi\|_X \leq 1} \|\eta\xi\|_{\ell_1}$ this is again a Banach sequence space that is symmetric whenever $X$ is so. Note that $(X_n)' = (X^\times)_n$ holds isometrically for each $n$. Following standard notation define the fundamental function of $X$ by

$$\lambda_X(n) := \|\sum_{k=1}^n e_k\|_X$$

for $n \in \mathbb{N}$, and recall that

$$\lambda_X(n)\lambda_{X^\times}(n) = n \tag{2}$$

(see, e.g., [14], 3.a.6). A Köthe function space $X$ is said to be $r$-convex (with $1 \leq r < \infty$) and $s$-concave (with $1 \leq s < \infty$) if there exists a constant $\kappa \geq 0$ such that for any choice $\xi_1, \ldots, \xi_n \in X$ we have

$$\Big\|\Big(\sum_{k=1}^n |\xi_k|^r\Big)^{1/r}\Big\|_X \leq \kappa\Big(\sum_{k=1}^n \|\xi_k\|_X^r\Big)^{1/r}$$

and

$$\Big(\sum_{k=1}^{n}\|\xi_k\|_X^s\Big)^{1/s}\le\kappa\Big\|\Big(\sum_{k=1}^{n}|\xi_k|^s\Big)^{1/s}\Big\|_X$$

respectively; we denote by $\mathbf{M}^{(r)}(X)$ and $\mathbf{M}_{(s)}(X)$ the smallest constants in each inequality. Recall that $X$ is $r$-convex ($s$-concave) if and only if $X^\times$ is $r'$-concave ($s'$-convex) (see [15], 1.d.4). If $X$ is $r$-convex for some $r$ or $s$-concave for some $s$, then we say that $X$ has non-trivial convexity or non-trivial concavity; 2-convex and 2-concave spaces will be of special interest. It is well known that $X$ is 2-concave if and only if it has cotype 2; in this case, $\mathbf{M}_{(2)}(X_n)\asymp\mathbf{C}_2(X_n)$. More generally, $\mathbf{M}_{(2)}(X_n)\asymp\mathbf{C}_2(X_n)$ whenever $X$ has non-trivial concavity (see [15], 1.d.6). Hence for $X$ with non-trivial concavity our conjecture from (1) can be reformulated as follows:

$$\mathbf{C}_2(\mathcal{P}(^mX_n))\asymp(n^{1/2})^{m-1}\mathbf{M}^{(2)}(X_n). \tag{3}$$

Let $E$ be any Banach space and $T\in\mathcal{L}(\ell_2^n;E)$. Then for independent Gaussian random variables $g_1,\dots,g_n$ on a probability space $(\Omega,\Sigma,\mu)$ the $l$-norm of $T$ is defined to be

$$l(T):=\Big(\int_\Omega\|\sum_j g_jT(e_j)\|_E^2\,d\mu\Big)^{1/2}.$$

This definition is independent of the choice of the orthonormal basis in $\ell_2^n$ and of the random variables $(g_j)_j$ (see, e.g., [23], Section 12).

We will frequently use the fact that if $X$ has non-trivial concavity, then

$$l(id\colon\ell_2^n\to X_n)\asymp\lambda_X(n); \tag{4}$$

indeed, from [23], (4.3), (4.7) and (4.8), we can take $r_1,\dots,r_n$ the classical Rademacher functions to get

$$l(\ell_2^n\to X_n)\asymp\int_\Omega\|\sum_{k=1}^n g_ke_k\|_X\,d\mu\asymp\int_0^1\|\sum_{k=1}^n r_k(t)e_k\|_X\,dt=\|\sum_{k=1}^n e_k\|_X.$$

As usual, define

$$X(E):=\{(x_i)_{i\in\mathcal{J}}\subseteq E\colon(\|x_i\|_E)_i\in X\},$$

where $X$ is a Köthe function space defined on a countable or finite set $\mathcal{J}$ as before and $E$ is any Banach space. Together with the norm $\|(x_i)_i\|_{X(E)}:=\|(\|x_i\|_E)_i\|_X$ this space becomes a Banach space. The natural embedding $X\otimes E\hookrightarrow X(E)$ induces a norm on the tensor product, and $X\otimes E$ endowed with this norm will be denoted by $X \otimes_X E$. It is known (see, e.g., [17], Lemma 1.12) that if $X$ is a 2-concave Banach sequence space and $E$ is any Banach space of cotype 2, then $X(E)$ has cotype 2 and

$$\mathbf{C}_2(X(E)) \leq \sqrt{2}\mathbf{M}_{(2)}(X)\mathbf{C}_2(E). \tag{5}$$

## 3. Reduction to full tensor products

We write $\otimes^m E$ for the $m$-th full tensor product of a Banach space $E$, and $\otimes^m_\varepsilon E$ whenever we endow this space with the injective norm $\varepsilon$. Similarly we denote by $\otimes^{m,s}_{\varepsilon_s} E$ the $m$-th symmetric tensor product of $E$ endowed with the symmetric injective norm $\varepsilon_s$. By the symmetrization map

$$S^m_E\colon \otimes^m E \to \otimes^m E, S^m_E(x_1 \otimes \cdots \otimes x_m) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)},$$

where $\Sigma_m$ stands for the group of permutations of $\{1, \ldots, m\}$, the space $\otimes^{m,s}_{\varepsilon_s} E$ can be considered as a complemented subspace of $\otimes^m_\varepsilon E$; recall that the natural embedding has norm $\leq m^m/m!$ and the projection $P = S^m_E$ has norm 1 (see, e.g., [10], 3.1). We use the notation $\otimes^m_2 \ell^n_2$ for the $m$-th Hilbert tensor product of $\ell^n_2$ (for $x = \sum_{i_1,\ldots,i_m} a_{(i_1,\ldots,i_m)} e_{i_1} \otimes \cdots e_{i_m}$, put $\|x\|_2 = (\sum_{i_1,\ldots,i_m} |a_{(i_1,\ldots,i_m)}|^2)^{1/2}$). It is well known that, if $M$ is a finite dimensional Banach space, we can represent the space of $m$-homogeneous polynomials on $M$ as the $m$-th symmetric tensor product of $M'$ (see [10], 5.3):

$$\otimes^{m,s}_{\varepsilon_s} M' = \mathcal{P}(^m M), \quad \otimes^m x' \mapsto [x \mapsto x'(x)^m]. \tag{6}$$

Let us see that in our case we can go a little bit further and even work in the full tensor product.

Proposition 3.1: *Let $X$ be a symmetric Banach sequence space and $m \in \mathbb{N}$. Let $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $a_{mn} \prec a_n$ (resp. $a_n \prec a_{mn}$); then the following are equivalent:*

(i) $\mathbf{C}_2(\mathcal{P}(^m X_n)) \prec a_n$ *(resp.* $a_n \prec \mathbf{C}_2(\mathcal{P}(^m X_n))$*).*
(ii) $\mathbf{C}_2(\otimes^{m,s}_{\varepsilon_s} X'_n) \prec a_n$ *(resp.* $a_n \prec \mathbf{C}_2(\otimes^{m,s}_{\varepsilon_s} X'_n)$*).*
(iii) $\mathbf{C}_2(\otimes^m_\varepsilon X'_n) \prec a_n$ *(resp.* $a_n \prec \mathbf{C}_2(\otimes^m_\varepsilon X'_n)$*).*

The proof is based on the following Lemma.

Lemma 3.2: *Let $X$ be a symmetric Banach sequence space and $m \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\otimes^m_\varepsilon X_n$ is a complemented subspace of $\otimes^{m,s}_{\varepsilon_s} X_{mn+k}$,*

$$\otimes^m_\varepsilon X_n \underset{P}{\overset{I}{\rightleftarrows}} \otimes^{m,s}_{\varepsilon_s} X_{mn+k},$$

*with* $\|I\| \leq 1$ *and* $\|P\| \leq m^m$.

*Proof:* Consider $n, k$ and, for each $i = 1, \dots, m$, let us define injections $I_i$: $X_n \to X_{mn+k}$ and projections $P_i$: $X_{mn+k} \to X_n$ by

$$I_i\Big(\sum_{j=1}^{n} \lambda_j e_j\Big) = \sum_{j=1}^{n} \lambda_j e_{n(i-1)+j}, \; P_i\Big(\sum_{j=1}^{mn+k} \lambda_j e_j\Big) = \sum_{j=1}^{n} \lambda_{n(i-1)+j} e_j.$$

Clearly, $I_i$ is an isometry and $\|P_i\| \leq 1$ for all $i$. Consider now the symmetrisation mapping $S^m_{X_{mn+k}}$ and the embedding $I^m_{X_{mn+k}}$: $\otimes^{m,s}_{\varepsilon_s} X_{mn+k} \to \otimes^m_\varepsilon X_{mn+k}$. Now we have, by [10], 1.10, that

$$\otimes^m_\varepsilon X_n \overset{\otimes I_i}{\to} \otimes^m_\varepsilon X_{mn+k} \overset{S^m_{X_{mn+k}}}{\to} \otimes^{m,s}_{\varepsilon_s} X_{mn+k} \overset{I^m_{X_{mn+k}}}{\to} \otimes^m_\varepsilon X_{mn+k} \overset{m!(\otimes P_i)}{\to} \otimes^m_\varepsilon X_n$$

gives the identity on $\otimes^m_\varepsilon X_n$. This clearly gives the conclusion. The bound on the norms comes from an immediate application of the metric mapping property of $\varepsilon$ and the above given norms for $S^m_{X_{mn+k}}$. ∎

*Proof of Proposition 3.1:* The equivalence (i) ⇔ (ii) follows in both cases immediately from the representation (6). For the proof of (ii) ⇔ (iii), let us assume first that $a_{mn} \prec a_n$. Then the implication (ii) ⇒ (iii) follows from Lemma 3.2,

$$\mathbf{C}_2(\otimes^m_\varepsilon X'_n) \leq m^m \mathbf{C}_2(\otimes^{m,s}_{\varepsilon_s} X'_{mn}) \prec a_{mn} \prec a_n.$$

On the other hand, (iii) ⇒ (ii) is immediate from the fact that $\otimes^{m,s}_{\varepsilon_s} X'_n \overset{\hookrightarrow}{\leftarrow} \otimes^m_\varepsilon X'_n$. Assume now that $a_{mn} \succ a_n$; then (ii) ⇒ (iii) follows from $\otimes^{m,s}_{\varepsilon_s} X'_n \overset{\hookrightarrow}{\leftarrow} \otimes^m_\varepsilon X'_n$.

To prove (iii) ⇒ (ii), let us define $[n/m] = \max\{k \in \mathbb{N}\colon k \leq n/m\}$ for each $n > m$. Then, from Lemma 3.2 we have that

$$\otimes^m_\varepsilon X_{[n/m]} \underset{P}{\overset{I}{\rightleftarrows}} \otimes^{m,s}_{\varepsilon_s} X_n,$$

with $\|I\| \leq 1$ and $\|P\| \leq m^m$. Then the proof goes as before. Notice that, depending on whether $m$ divides $n$ or not, we have that $n = [n/m]m$ or $n = [n/m]m + 1$. ∎

*Remark 3.3:* The sequence $a_n = (n^{1/2})^{m-1}\mathbf{C}_2(X'_n)$ considered in our conjecture satisfies that for any fixed $m$, $(a_{nm}) \asymp (a_n)$. Indeed, we obviously have that $(n^{1/2})^{m-1} \asymp ((nm)^{1/2})^{m-1}$. For the cotype 2 constants note that $X'_n = (X^\times)_n$

and $\mathbf{C}_2(Y_n) \asymp \mathbf{C}_2(Y_{nm})$ for each $m$ and each symmetric sequence space $Y$; indeed, if we factorize as follows, using the mappings defined in the proof of Lemma 3.2,

$$\begin{array}{ccc} Y_{mn} & \longrightarrow & Y_{mn} \\ {\scriptstyle P_i}\downarrow & & \uparrow{\scriptstyle I_i} \\ Y_n & \longrightarrow & Y_n \end{array}$$

then $id_{Y_{mn}} = \sum_{i=1}^{m} I_i id_{Y_n} P_i$ and hence $\mathbf{C}_2(id_{Y_n}) \le \mathbf{C}_2(id_{Y_{mn}}) \le m\mathbf{C}_2(id_{Y_n})$. This altogether proves that a symmetric Banach sequence space $X$ satisfies our conjecture (1) (for $X$ with non-trivial concavity see also (3)) if and only if

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}\mathbf{C}_2(X_n). \tag{7}$$

With this, we can give a first positive answer: Since $\mathbf{C}_2(\ell_\infty^n) \asymp n^{1/2}(\log(n+1))^{-1/2}$ ([23], Section 4) and $\otimes_\varepsilon^m \ell_\infty^n = \ell_\infty^{n^m}$, we get

$$\mathbf{C}_2(\otimes_\varepsilon^m \ell_\infty^n) \asymp \frac{(n^m)^{1/2}}{\sqrt{\log(n+1)}} \asymp (n^{1/2})^{m-1}\mathbf{C}_2(\ell_\infty^n) \asymp \mathbf{C}_2(\mathcal{P}(^m\ell_1^n)). \tag{8}$$

## 4. A general estimate

Let us give now a general estimate, weaker than our conjecture.

Theorem 4.1: *Let $X$ be any symmetric Banach sequence space and $m \in \mathbb{N}$; then,*

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^m. \tag{9}$$

*Moreover, if $X$ has non-trivial convexity,*

$$(n^{1/2})^{m-1} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^m. \tag{10}$$

The upper estimate is trivial: Note that for any $n$-dimensional Banach space $E_n$ we have that

$$\mathbf{C}_2(E_n) \le d(E_n, \ell_2^n)\mathbf{C}_2(\ell_2^n) \le \sqrt{n},$$

where $d$ stands as usual for the Banach-Mazur distance. Hence, since $\otimes_\varepsilon^m X_n$ has dimension $n^m$, the upper estimate follows.

For the proof of the lower estimate for $\mathbf{C}_2(\otimes_\varepsilon^m X_n)$ we need some work. Let us recall first that if $E, F$ are Banach spaces and $T \in \mathcal{L}(E;F)$, then for each $k \in \mathbb{N}$ the $k$-th approximation number of $T$ is defined to be

$$a_k(T) = \inf\{\|T - S\|: S \in \mathcal{L}(E;F),\ \operatorname{rank} S < k\}.$$

For a complete study of these approximation numbers in the more general frame of the $s$-numbers see, e.g., [13], [18], [19].

Our starting point is the following result of Pisier (see, e.g., [20], Chapter 10): For any Banach space $E$ of cotype 2, every $T \in \mathcal{L}(\ell_2^n; E)$,

$$k^{1/2} a_k(T) \prec \mathbf{C}_2(E)\; l(T). \tag{11}$$

In particular, for any Banach sequence space $X$ and any $m$,

$$k^{1/2}\; a_k(id\colon \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n)\; l(id\colon \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n). \tag{12}$$

Our goal now is to try to estimate in (12) the approximation number from below and the $l$-norm from above in order to get a bound which can be evaluated in concrete cases. Let us begin with the approximation numbers. Suppose that $E_n$ equals $\mathbb{R}^n$ with some norm such that there is a group $G$ of linear bijections on $X_n$ satisfying:

(i) All $g \in G$, when considered as operators on $\ell_2^n$, are isometries.

(ii) Given a linear mapping $u\colon E_n \to E_n$, if $ug = gu$ for all $g \in G$, then there is $\lambda \in \mathbb{R}$ such that $u = \lambda id_{E_n}$.

(iii) For each $u \in \mathcal{L}(E_n)$ and all $g_1, g_2 \in G$, we have that $\|g_1 u g_2\| = \|u\|$.

In [6] it is proved that if $E_n$ is such a space, then

$$a_{[n/2]}(id\colon \ell_2^n \to E_n) \asymp \|id\colon \ell_2^n \to E_n\|. \tag{13}$$

Let now $X$ be any symmetric Banach sequence space and let us fix $n \in \mathbb{N}$. For each choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$ and any permutation $\pi \in \Sigma_n$ we can define mappings

$$M_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n, \quad T_\pi\colon \mathbb{R}^n \to \mathbb{R}^n$$

just by letting $M_\varepsilon$ multiply each coordinate by the corresponding $\varepsilon_k$ and $T_\pi$ reorder the tuple according to $\pi$. Each one of these mappings is obviously an isometry on $X_n$. Define now $S(\mathbb{R}^n)$ to be the group generated by the set

$$\{M_\varepsilon\colon \varepsilon \in \{-1, +1\}^n\} \cup \{T_\pi\colon \pi \in \Sigma_n\},$$

and

$$S(\otimes^m \mathbb{R}^n) := \{T_1 \otimes \cdots \otimes T_m\colon T_j \in S(\mathbb{R}^n),\ j = 1, \dots, m\}.$$

Obviously, $S(\mathbb{R}^n)$ considered as linear bijections on $X_n$ has the properties (i)–(iii) stated above and, as a consequence, (13) holds for $E_n = X_n$; let us see now that, using $S(\otimes^m \mathbb{R}^n)$, we can even establish (13) for $E_n = \otimes_\varepsilon^m X_n$.

PROPOSITION 4.2: *Let $X$ be a symmetric Banach sequence space and $m \in \mathbb{N}$. Then,*

$$a_{[n^m/2]}(id\colon \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) \asymp \|id\colon \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|.$$

For $m = 2$ this result was proved in [1], Corollary in Section 3; we were unable to extend by induction the proof given there so that we present an alternative approach.

*Proof:* Let us begin by showing that $S(\otimes^m \mathbb{R}^n)$, considered as a group of bijections on $\otimes_\varepsilon^m X_n$, satisfies analogous properties to the ones given in (i)–(iii). Firstly, each operator in $S(\otimes^m \mathbb{R}^n)$ defines an isometry on $\otimes_2^m \ell_2^n$ since each $T \in S(\mathbb{R}^n)$ is an isometry on $\ell_2^n$ and, hence, $(T_1(e_{j_1})\otimes\cdots\otimes T_m(e_{j_m}))_{j_1,\ldots,j_m=1,\ldots,n}$ is an orthonormal basis of $\otimes_2^m \ell_2^n$.

For the proof of the analog of (ii) we use an argument that was first used by Gordon and Lewis in [11]. We proceed by induction on $m$. It is known that the conclusion holds for $m = 1$; suppose now that it also holds for $\otimes^{m-1} X_n$. Clearly, the following set equals $S(\otimes^m \mathbb{R}^n)$:

$$\{\tilde{T} \otimes T_m\colon \tilde{T} \in S(\otimes^{m-1}\mathbb{R}^n),\ T_m \in S(\mathbb{R}^n)\}.$$

Consider now $u\colon \otimes^m X_n \to \otimes^m X_n$ such that $Tu = uT$ for all $T \in S(\otimes^m \mathbb{R}^n)$. Fix $\xi \in X_n$ and $\xi^* \in X_n^*$ and define the mapping

$$v\colon \otimes^{m-1} X_n \to \left(\otimes^{m-1} X_n\right)^{**} = \otimes^{m-1} X_n$$

by $\langle v(\eta), \eta^*\rangle = \langle u(\eta \otimes \xi), \eta^* \otimes \xi^*\rangle$. Then $\tilde{T}v = v\tilde{T}$ for all $\tilde{T} \in S(\otimes^{m-1} X_n)$; indeed, for all $\eta \in \otimes^{m-1} X_n$ and $\eta^* \in \left(\otimes^{m-1} X_n\right)^*$ we have

$$\langle v\tilde{T}\eta, \eta^*\rangle = \langle \tilde{T}v\eta, \eta^*\rangle.$$

By assumption there is a unique scalar $\lambda(\xi, \xi^*)$ such that $\langle v\eta, \eta^*\rangle = \lambda(\xi,\xi^*)\langle \eta, \eta^*\rangle$ for all $\eta \in \otimes^{m-1} X_n$ and $\eta^* \in \left(\otimes^{m-1} X_n\right)^*$. Now define

$$w\colon X_n \to X_n^{**} = X_n, \quad \langle w\xi, \xi^*\rangle = \lambda(\xi, \xi^*).$$

Consider $\eta_0 \in \otimes^{m-1} X_n$ and $\eta_0^* \in \otimes^{m-1} X_n^*$ such that $\langle \eta_0, \eta_0^*\rangle = 1$; then $\lambda(\xi, \xi^*) = \langle v\eta_0, \eta_0^*\rangle = \langle u(\eta_0 \otimes \xi), \eta_0^* \otimes \xi^*\rangle$. With this, proceeding in the same way as before we have that $wT_m = T_m w$ for all $T_m \in S(\mathbb{R}^n)$. Therefore, we can find $t$ such that $t\langle \xi, \xi^*\rangle = \langle w\xi, \xi^*\rangle = \lambda(\xi, \xi^*)$ for all $\xi \in X_n$ and $\xi^* \in X_n^*$. Hence for every $\eta, \xi, \eta^*, \xi^*$,

$$\langle u(\eta \otimes \xi), \eta^* \otimes \xi^*\rangle = t\langle \xi, \xi^*\rangle\langle \eta, \eta^*\rangle = t\langle \eta \otimes \xi, \eta^* \otimes \xi^*\rangle,$$

which finally shows that $u = t\, id_{\otimes^m X_n}$.

The third condition follows obviously from the fact that $S(\otimes^m \mathbb{R}^n)$ consists of isometries on $\otimes_\varepsilon^m \mathbb{R}^n$.

Take now some linear isomorphism $\Psi\colon \otimes^m \mathbb{R}^n \to \mathbb{R}^{n^m}$ such that, when the Hilbert norms are considered, $\Psi\colon \otimes_2^m \ell_2^n \to \ell_2^{n^m}$ is an isometry. With this we can define a norm in $\mathbb{R}^{n^m}$ by putting $\|\xi\|_\varepsilon := \varepsilon(\Psi^{-1}(\xi))$. In this way, clearly,

$$\Psi\colon \otimes_\varepsilon^m X_n \to (\mathbb{R}^{n^m}, \|\ \|_\varepsilon)$$

is an isometry. Define now the group $\Sigma(\mathbb{R}^{n^m}) := \{\Psi T \Psi^{-1}\colon T \in S(\otimes^m \mathbb{R}^n)\}$. Each $R \in \Sigma(\mathbb{R}^{n^m})$ obviously is an isometry on $\ell_2^{n^m}$. On the other hand, if $u\colon \mathbb{R}^{n^m} \to \mathbb{R}^{n^m}$ is linear and such that $uR = Ru$ for all $R \in \Sigma(\mathbb{R}^{n^m})$, then $\Psi^{-1} u \Psi T = T \Psi^{-1} u \Psi$ for all $T \in S(\otimes^m \mathbb{R}^n)$. Using what we just proved for $S(\otimes^m \mathbb{R}^n)$ we get that there is some $\lambda$ so that $\Psi^{-1} u \Psi = \lambda id_{\otimes^m X_m}$ and, hence, $u = \lambda id_{\mathbb{R}^{n^m}}$.

Given any two $R_1, R_2 \in \Sigma(\mathbb{R}^{n^m})$, since they are isometries on $(\mathbb{R}^{n^m}, \|\ \|_\varepsilon)$, for all $u \in \mathcal{L}(\mathbb{R}^{n^m}, \|\ \|_\varepsilon)$ we have that $\|R_1 u R_2\| = \|u\|$.

Hence $\Sigma(\mathbb{R}^{n^m})$, considered as a group of linear bijections on $(\mathbb{R}^{n^m}, \|\ \|_\varepsilon)$, satisfies the above conditions (i)–(iii) so that we finally can apply (13) and the fact that $\Psi$ is an isometry to obtain

$$\begin{aligned} &a_{[n^m/2]}(\otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) = a_{[n^m/2]}((\mathbb{R}^{n^m}, \|\ \|_2) \to (\mathbb{R}^{n^m}, \|\ \|_\varepsilon)) \\ \asymp &\|(\mathbb{R}^{n^m}, \|\ \|_2) \to (\mathbb{R}^{n^m}, \|\ \|_\varepsilon)\| = \| \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|. \qquad \blacksquare \end{aligned}$$

In order to handle the $l$-norm on the right-hand side of (12) recall Chevét's inequality (see, e.g., [23], (43.2)): For any two operators $T \in \mathcal{L}(\ell_2^n; E)$ and $S \in \mathcal{L}(\ell_2^m; F)$,

$$l(T \otimes S\colon \ell_2^n \otimes_2 \ell_2^m \to E \otimes_\varepsilon F) \le c(\|T\| l(S) + l(T)\|S\|), \tag{14}$$

where $c \ge 0$ is some universal constant. In [3], Lemma 6, it is shown by induction of Chevét's inequality that for $T \in \mathcal{L}(\ell_2^n; E)$ and $m \ge 2$,

$$l(\otimes^m T\colon \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m E) \le c_m l(T) \|T\|^{m-1}, \tag{15}$$

where $c_m \ge 0$ is a constant depending only on $m$.

Lemma 4.3: *Let $X$ be a symmetric Banach sequence space and $m \in \mathbb{N}$; then*

$$(n^{1/2})^m \frac{\|id\colon \ell_2^n \to X_n\|}{l(id\colon \ell_2^n \to X_n)} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n). \tag{16}$$

*Proof:* Putting $k = [n^m/2]$ in (12) and using Proposition 4.2 we get

$$(17) \qquad (n^m)^{1/2} \| \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n \| \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n)\; l(\otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n).$$

To finish, note that $\|\ell_2^n \to X_n\|^m = \| \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|$ and use (15). ∎

With this we can now give the

*Proof of Theorem 4.1:* The upper estimate has already been proved. The lower one will follow from Lemma 4.3 applied to the symmetric Banach sequence space $X^\times$ (instead of $X$). To begin with, from the definition and applying [23], (4.4) and (4.8) we have

$$\begin{aligned} l(\ell_2^n \to X_n') &= \Big( \int_\Omega \| \sum_{k=1}^n g_k e_k \|_{X^\times}^2 d\mu \Big)^{1/2} \\ &\le M \sqrt{\log(n+1)} \Big( \int_0^1 \| \sum_{k=1}^n r_k(t) e_k \|_{X^\times}^2 dt \Big)^{1/2} \\ &= M \sqrt{\log(n+1)} \Big\| \sum_{k=1}^n e_k \Big\|_{X^\times}. \end{aligned}$$

On the other hand, $\| \sum_{k=1}^n e_k \|_{X_n'} \le \sqrt{n} \|\ell_2^n \to X_n'\|$. From this and (16) we obtain

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n') \succ \frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}}.$$

With this and Proposition 3.1 we get (9). For (10), notice that if $X$ has non-trivial convexity, then $X^\times$ has non-trivial concavity, and hence by (4), $l(\ell_2^n \to X_n') \asymp \lambda_{X^\times}(n)$. ∎

## 5. Proof of the conjecture for 2-convex spaces

Let us prove now that our conjecture (1) (see also (3)) is true when $X$ is a 2-convex symmetric Banach sequence space.

THEOREM 5.1: *Let $X$ be a symmetric 2-convex Banach sequence space and fix $m$; then*

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp (n^{1/2})^{m-1}.$$

Three lemmas are needed for the proof. The following definition was introduced in [16] and is a generalization of the classical concept of $(p,q)$-summing operators.

*Definition 5.2:* Let $X, Y$ be two Köthe function spaces modeled on some finite or countable set $\mathcal{J}$ and $E, F$ two Banach spaces; we say that $T \in \mathcal{L}(E;F)$ is $(Y,X)$-summing if there exists some constant $c \geq 0$ such that for any finite $\mathcal{I} \subseteq \mathcal{J}$ and $(x_i)_{i\in\mathcal{I}} \subseteq E$, we have that

$$\|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_Y \leq c \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_X;$$

the best constant in this inequality is denoted by $\pi_{Y,X}(T)$.

For $X = \ell_p$ we as usual write $\pi_p = \pi_{X,X}$. It is easily shown that if $i\colon X \hookrightarrow Y$, then $T \in \mathcal{L}(E;F)$ is $(Y,X)$-summing if and only if $i \otimes T\colon X \otimes_\varepsilon E \to Y \otimes_Y F$ is continuous and, in this case,

$$\pi_{Y,X}(T) = \|i \otimes T\colon X \otimes_\varepsilon E \to Y \otimes_Y F\|.$$

The following result is a slight modification of Lemma 1.2 in [16]; for the sake of completeness we give here an adapted proof.

LEMMA 5.3: *Let $Y$ be a Köthe function space modeled on $\mathcal{J}$, finite or countable, such that $Y = Y^{\times\times}$; then any $(\ell_1(\mathcal{J}), \ell_1(\mathcal{J}))$-summing operator $T \in \mathcal{L}(E;F)$ is $(Y,Y)$-summing and*

$$\pi_{Y,Y}(T) \leq \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T).$$

*Proof:* Take $\mathcal{I} \subseteq \mathcal{J}$ finite and $(x_i)_{i\in\mathcal{I}} \subseteq E$; we know that

$$\|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})} \leq \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})}.$$

Since $Y = Y^{\times\times}$,

$$\begin{aligned}
\|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_Y &= \|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_{Y^{\times\times}} = \sup_{\|(\zeta_i)_i\|_{Y^\times}\leq 1} \|(\zeta_i\|Tx_i\|_F)_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})} \\
&\leq \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T) \sup_{\|(\zeta_i)_i\|_{Y^\times}\leq 1} \sup_{\|x'\|_{E'}\leq 1} \|(x'(\zeta_i x_i))_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})} \\
&= \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \sup_{\|(\zeta_i)_i\|_{Y^\times}\leq 1} \|(\zeta_i x'(x_i))_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})} \\
&= \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_{Y^{\times\times}} \\
&= \pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_{Y}. \qquad \blacksquare
\end{aligned}$$

The assumption $Y = Y^{\times\times}$ is equivalent to the Fatou property of $Y$ (see [15], Section 1.b).

The following Lemma is of independent interest.

LEMMA 5.4: *Let $Y$ be as in Lemma 5.3; then for each Banach sequence space $X$ and $n \in \mathbb{N}$*

$$\pi_{Y,Y}(id_{X_n}) = \|id: Y \otimes_\varepsilon X_n \to Y \otimes_Y X_n\| \leq K_G \mathbf{M}_{(2)}(X_n)\ n^{1/2},$$

*where $K_G \geq 1$ is Grothendieck's constant.*

*Proof:* By the Grothendieck-Krivine inequality (see [15], Theorem 1.f.14), for any $T \in \mathcal{L}(\mathcal{C}(K); X_n)$ and finitely many $x_1, \ldots, x_m \in \mathcal{C}(K)$,

$$\left\| \left( \sum_{k=1}^m |Tx_k|^2 \right)^{1/2} \right\|_X \leq K_G \|T\| \left\| \left( \sum_{k=1}^m |x_k|^2 \right)^{1/2} \right\|_{\mathcal{C}(K)}.$$

Hence,

$$\begin{aligned} \left( \sum_{k=1}^m \|Tx_k\|_X^2 \right)^{1/2} &\leq \mathbf{M}_{(2)}(X_n)\ \| \left( \sum_{k=1}^m |Tx_k|^2 \right)^{1/2} \|_X \\ &\leq K_G \mathbf{M}_{(2)}(X_n)\ \|T\|\ \| \left( \sum_{k=1}^m |x_k|^2 \right)^{1/2} \|_{\mathcal{C}(K)} \\ &= K_G \mathbf{M}_{(2)}(X_n)\ \|T\| \sup_{\|x'\| \leq 1} \left( \sum_{k=1}^m |x'(x_k)|^2 \right)^{1/2}, \end{aligned}$$

so that $\pi_2(T) \leq K_G \mathbf{M}_{(2)}(X_n) \|T\|$. Then, by [23], Proposition 10.17, and the well known fact that $\pi_2(id_{X_n}) = \sqrt{n}$,

$$\pi_1(id_{X_n}) \leq K_G \mathbf{M}_{(2)}(X_n) \pi_2(id_{X_n}) = K_G \mathbf{M}_{(2)}(X_n) \sqrt{n}.$$

With this, applying Lemma 5.3, we get

$$\begin{aligned} \|Y \otimes_\varepsilon X_n \to Y \otimes_Y X_n\| &= \pi_{Y,Y}(id_{X_n}) \leq \pi_{\ell_1,\ell_1}(id_{X_n}) \\ &= \pi_1(id_{X_n}) \leq K_G \mathbf{M}_{(2)}(X_n) \sqrt{n}. \qquad \blacksquare \end{aligned}$$

For the formulation of the third Lemma we need some further notation. If $X$ is a Banach sequence space let us define $[X]^2$ to be the space of functions $\phi: \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that $(\phi(n,k))_k \in X$ for all $n$ and $(\|(\phi(n,k))_k\|_X)_n \in X$. Define the norm $\|\phi\|_{[X]^2} = \|(\|(\phi(n,k))_k\|_X)_n\|_X$, and note that $[X]^2$ together with this norm is a Köthe function space modeled on $\mathbb{N}^2$. Notice that $[X]^2 \cong X(X)$ holds isometrically.

Suppose that $[X]^{m-1}$ has been defined and let us define $[X]^m$ like those functions $\phi: \mathbb{N}^m \to \mathbb{R}$ such that $(\phi(i_1, \ldots, i_{m-1}, i_m))_{i_m \in \mathbb{N}} \in X$ for all $(i_1, \ldots, i_{m-1}) \in$

$\mathbb{N}^{m-1}$ and $(\|(\phi(i_1,\dots,i_{m-1},i_m))_{i_m}\|_X)_{(i_1,\dots,i_{m-1})} \in [X]^{m-1}$. We endow it with the norm

$$\|\phi\|_{[X]^m} = \|(\|(\phi(i_1,\dots,i_{m-1},i_m))_{i_m}\|_X)_{(i_1,\dots,i_{m-1})}\|_{[X]^{m-1}}.$$

With this norm $[X]^m$ is a Köthe function space modeled on $\mathbb{N}^m$ and $[X]^m \cong [X]^{m-1}(X)$. A simple iteration of (5) shows that for each $m$ there is some constant $K > 0$ such that, for each 2-concave Banach sequence space $X$,

$$\mathbf{C}_2([X]^m) \le K\, \mathbf{M}_{(2)}(X)^m. \tag{18}$$

Lemma 5.5: *Let $X$ be a Banach sequence space and $m \in \mathbb{N}$; then, for all $n$,*

$$\|id\colon \otimes_\varepsilon^m X_n \to [X_n]^m\| \le K_G^{m-1}\mathbf{M}_{(2)}(X_n)^{m-1}\,(n^{1/2})^{m-1}.$$

Note that when $X$ is 2-concave, $\mathbf{M}_{(2)}(X_n) \le \mathbf{M}_{(2)}(X)$ for all $n$.

*Proof:* Let us prove it by induction. The case $m = 2$ follows from Lemma 5.4 (put $Y = X_n$). Suppose that the result holds for $m-1$. Considering the following commutative diagram with the natural ma:pings

$$\begin{array}{ccc} \otimes_\varepsilon^m X_n = (\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n & \longrightarrow & [X_n]^m = [X_n]^{m-1}(X_n) \\ \downarrow & \nearrow & \\ [X_n]^{m-1} \otimes_\varepsilon X_n & & \end{array}$$

we get

$$\begin{aligned} \|\otimes_\varepsilon^m X_n \to [X_n]^m\| \le & \|(\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_\varepsilon X_n\| \cdot \\ & \|[X_n]^{m-1} \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_{[X_n]^{m-1}} X_n\|. \end{aligned}$$

From Lemma 5.4, we know that $\|[X_n]^{m-1} \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_{[X_n]^{m-1}} X_n\| \le K_G \mathbf{M}_{(2)}(X_n)\, n^{1/2}$. As $\|(\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_\varepsilon X_n\| \le \|\otimes_\varepsilon^{m-1} X_n \to [X_n]^{m-1}\|$, by applying the induction hypothesis we obtain

$$\|\otimes_\varepsilon^m X_n \to [X_n]^m\| \le K_G^{m-1}\mathbf{M}_{(2)}(X_n)^{m-1}\,(n^{1/2})^{m-1}. \qquad \blacksquare$$

With all this we are now ready to give the

*Proof of Theorem 5.1:* Notice first of all that it is enough to prove that every $X$, 2-concave and symmetric, satisfies $\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}$. Indeed, taking $X$ a 2-convex space and applying this to $X_n'$ jointly with Proposition 3.1 we get

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp (n^{1/2})^{m-1}.$$

Let then $X$ be any 2-concave symmetric Banach sequence space. To get the desired upper estimate of $\mathbf{C}_2(\otimes_\varepsilon^m X_n)$ let us factorize

$$\begin{array}{ccc} \otimes_\varepsilon^m X_n & \xrightarrow{\quad id \quad} & \otimes_\varepsilon^m X_n \\ & \searrow \quad \nearrow & \\ & [X_n]^m & \end{array}$$

Then, from Lemma 5.5 and the fact that $\|[X_n]^m \to \otimes_\varepsilon^m X_n\| \le 1$ we have that

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^{m-1}\mathbf{C}_2([X_n]^m).$$

Now, by (18), $\mathbf{C}_2([X_n]^m) \prec \mathbf{M}_{(2)}(X_n)^m \le \mathbf{M}_{(2)}(X)^m$, hence

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^{m-1}.$$

This, together with Proposition 3.1, gives the upper estimate. The lower bound follows from Theorem 4.1, since $X$ is non-trivially convex. ∎

## 6. Proof of the conjecture for 2-concave spaces

Let us now see how our conjecture (1), and also (3) follows for 2-concave spaces with non-trivial convexity, giving the following

THEOREM 6.1: *Let $X$ be a 2-concave symmetric Banach sequence space with non-trivial convexity and fix $m \in \mathbb{N}$; then*

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp (n^{1/2})^{m-1}\mathbf{M}^{(2)}(X_n) \asymp n^{m/2-1}\lambda_X(n).$$

As we did in the previous section, we are going to prove the theorem using Proposition 3.1, which means to prove that for every 2-convex $X$ with non-trivial concavity

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}\mathbf{M}_{(2)}(X_n) \asymp \frac{(n^{1/2})^m}{\lambda_X(n)}; \tag{19}$$

indeed, if this is true and $X$ is 2-concave with non-trivial convexity we can apply it to its Köthe dual and use (2) to obtain the result. We start with the proof of the second estimate in (19) for $m = 1$.

LEMMA 6.2: *Let $X$ be a 2-convex symmetric Banach sequence space with non-trivial concavity; then,*

$$\mathbf{M}_{(2)}(X_n) \asymp \frac{n^{1/2}}{\lambda_X(n)}.$$

*Proof:* For the upper bound we factorize the identity as usual,

$$\begin{array}{ccc} X_n & \longrightarrow & X_n \\ \big\downarrow & & \big\uparrow \\ \ell_2^n & \longrightarrow & \ell_2^n \end{array}$$

Note first that if $X$ is 2-convex, then $1 \leq \|\ell_2^n \to X_n\| \leq \mathbf{M}^{(2)}(X)$ for all $n$. Clearly from this we obtain

$$\mathbf{M}_{(2)}(X_n) \leq \|X_n \to \ell_2^n\| \, \mathbf{M}_{(2)}(\ell_2^n) \, \|\ell_2^n \to X_n\| \leq \|\ell_2^n \to X_n'\| \, \mathbf{M}^{(2)}(X).$$

Now, since $X$ is 2-convex, $X^\times$ is 2-concave and we have already noted $X_n' = (X^\times)_n$. Moreover, by [22], Proposition 2.2 (see also [5]), we have $\|\ell_2^n \to E_n\| \asymp n^{-1/2}\lambda_{E_n}(n)$ whenever $E$ is a 2-concave Banach sequence space. With all this and (2),

$$\mathbf{M}_{(2)}(X_n) \leq \mathbf{M}^{(2)}(X)\|\ell_2^n \to X_n'\| \prec \frac{n^{1/2}}{\lambda_X(n)}.$$

For the lower estimate take $k = [n/2]$ in (11) to get

$$[n/2]^{1/2} a_{[n/2]}(\ell_2^n \to X_n) \prec \mathbf{M}_{(2)}(X_n) l(\ell_2^n \to X_n).$$

Now, since $X$ has non-trivial concavity, we can apply (4) and (13) to get

$$\mathbf{M}_{(2)}(X_n) \succ \frac{\|\ell_2^n \to X_n\|}{\lambda_X(n)} n^{1/2} \geq \frac{n^{1/2}}{\lambda_X(n)}.$$

This completes the proof. ∎

The next lemma corresponds to Lemma 5.4.

LEMMA 6.3: *Let $X$ be a Banach sequence space and $Y$ be a Köthe function space modeled on some finite or countable set; then, for every $n$,*

$$\|id\colon X_n \otimes_\varepsilon Y \to X_n(Y)\| \leq \lambda_X(n).$$

*Proof:* We factorize in the following way,

$$\begin{array}{ccc} X_n \otimes_\varepsilon Y & \longrightarrow & X_n(Y) \\ & \searrow \quad \nearrow & \\ & \ell_\infty^n \otimes_\varepsilon Y = \ell_\infty^n(Y) & \end{array}$$

and get $\|X_n \otimes_\varepsilon Y \to X_n(Y)\| \leq \|\ell_\infty^n \otimes_\varepsilon Y \to X_n(Y)\|$. Let us estimate this latter norm: For $\zeta_1, \dots, \zeta_n \in Y$,

$$\begin{aligned}\|\sum_{k=1}^n e_k \otimes \zeta_k\|_{X_n(Y)} &= \|(\|\zeta_k\|_Y)_{k=1}^n\|_{X_n} \leq \|\ell_\infty^n \to X_n\| \ \sup_k \|\zeta_k\|_Y \\ &= \|\ell_\infty^n \to X_n\| \ \|\sum_{k=1}^n e_k \otimes \zeta_k\|_{\ell_\infty^n(Y)},\end{aligned}$$

hence

$$\|\ell_\infty^n \otimes_\varepsilon Y \to X_n(Y)\| \leq \|\ell_\infty^n \to X_n\| = \sup_{|\lambda_k|\leq 1} \|\sum_{k=1}^n \lambda_k e_k\|_{X_n} \leq \|\sum_{k=1}^n e_k\|_X,$$

which proves our claim. ∎

Analogously to what we did before Lemma 5.5, we define $[X]_2$ as those functions $\phi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ such that $(\phi(n,k))_n \in X$ for all $k$ and $(\|(\phi(n,k))_n\|_X)_k \in X$; with the norm defined in the obvious way we have that $[X]_2 \cong X(X)$ holds isometrically. Let us suppose that $[X]_{m-1}$ has been defined and define $[X]_m$ to be the space of all $\phi\colon \mathbb{N}^m \to \mathbb{R}$ such that $(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1})\in\mathbb{N}^{m-1}} \in [X]_{m-1}$ for all $i_m$ and $(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1})}\|_{[X]_{m-1}})_{i_m} \in X$. We define the norm

$$\|\phi\|_{[X]_m} = \|(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1})}\|_{[X]_{m-1}})_{i_m}\|_X.$$

In this case, we have that $[X]_m \cong X([X]_{m-1})$ holds isometrically. Once again, a repeated application of (5) gives that for each $m$ there is some constant $K > 0$ such that, for each 2-concave $X$,

$$\mathbf{C}_2([X]_m) \leq K\ \mathbf{M}_{(2)}(X)^m. \tag{20}$$

LEMMA 6.4: *Let $X$ be any Banach sequence space and $m \in \mathbb{N}$; then for all $n$,*

$$\|id\colon \otimes_\varepsilon^m X_n \to [X_n]_m\| \leq \lambda_X(n)^{m-2}.$$

*Proof:* We prove it by induction; the case $m = 2$ is clear from Lemma 6.3. Assume the result to be true for $m-1$; that is, $\|\otimes_\varepsilon^{m-1} X_n \to [X_n]_{m-1}\| \leq \lambda_X(n)^{m-2}$. For the $m$-th case we have

$$\begin{array}{ccc}\otimes_\varepsilon^m X_n = X_n \otimes_\varepsilon (\otimes_\varepsilon^{m-1} X_n) & \longrightarrow & [X_n]_m = X_n([X_n]_{m-1}) \\ \big\downarrow & \nearrow & \\ X_n \otimes_\varepsilon [X_n]_{m-1} & & \end{array}$$

The metric mapping property and the induction hypothesis imply

$$\|X_n \otimes_\varepsilon (\otimes_\varepsilon^{m-1} X_n) \to X_n \otimes_\varepsilon [X_n]_{m-1}\| \leq \| \otimes_\varepsilon^{m-1} X_n \to [X_n]_{m-1}\| \leq \lambda_X(n)^{m-2}.$$

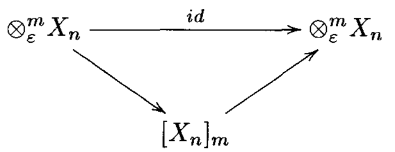
By Lemma 6.3, $\|X_n \otimes_\varepsilon [X]_{m-1} \to X_n([X]_{m-1})\| \leq \lambda_X(n)$, thus

$$\| \otimes_\varepsilon^m X_n \to [X_n]_m\| \leq \lambda_X(n)^{m-1}. \qquad ∎$$

*Proof of Theorem 6.2:* Recall that it is enough to show (19) for 2-convex spaces with non-trivial concavity; in fact, in view of Lemma 6.2 we just prove

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n).$$

First of all, in order to check the upper estimate let us again factorize,




Since $\|[X_n]_m \to \otimes_\varepsilon^m X_n\| \leq 1$, together with (20), Lemma 6.4 and Lemma 6.2 we get

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec \lambda_X(n)^{m-1} \mathbf{M}_{(2)}(X_n)^m \prec (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n).$$

For the lower estimate we start from (16). Since $X$ has non-trivial concavity, using Lemma 6.2 and (2), we have that

$$l(id\colon \ell_2^n \to X_n) \asymp \lambda_X(n) \asymp n^{1/2} \mathbf{M}_{(2)}(X_n)^{-1}.$$

From this and the fact that $\|\ell_2^n \to X_n\| \geq 1$, we finally obtain

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \succ (n^{1/2})^m \frac{\|\ell_2^n \to X_n\|}{l(\ell_2^n \to X_n)} \succ (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n). \qquad ∎$$

## 7. The complex case

Our main results on conjecture (1) (Theorems 4.1, 5.1, 6.1) can be easily carried over to complex Banach sequence spaces (defined in the obvious way). For a complex Banach sequence space $X$ define

$$X(\mathbb{R}) := \{y \in X\colon y_n \in \mathbb{R} \text{ for all } n\}$$

and endow it with the induced norm from $X$. Then $X(\mathbb{R})$ is a real Banach sequence space which is symmetric or 2-convex or 2-concave if $X$ itself is (with the obvious definitions).

PROPOSITION 7.1: *Let $X$ be a complex symmetric Banach sequence space. Then for each $m$,*

$$\mathbf{C}_2(\mathcal{P}(^mX(\mathbb{R})_n)) \prec \mathbf{C}_2(\mathcal{P}(^mX_n)) \prec \mathbf{C}_2(\mathcal{P}(^mX(\mathbb{R})_{2n})). \tag{21}$$

*In particular, if $(a_n) \asymp (a_{2n})$ and $(b_n) \asymp (b_{2n})$, then*

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^mX(\mathbb{R})_n)) \prec (b_n)$$

*if and only if*

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^mX_n)) \prec (b_n).$$

We start the proof with some remarks of independent interest.

*Remark 7.2:* For any complex Banach space $Z$ denote the underlying real Banach space by $Z_{\mathbb{R}}$; let us observe that in this situation $\mathcal{P}(^mZ_{\mathbb{R}})$ denotes the real Banach space of continuous real $m$-homogeneous polynomials from $Z_{\mathbb{R}}$ into $\mathbb{R}$. Let now $(Y, \| \ \|)$ be a real Banach space, and let us consider an arbitrary complexification $(Y+iY, \| \ \|_{\mathbb{C}})$ of $(Y, \| \ \|)$ (i.e., $Y+iY = Y\times Y$ together with a norm that satisfies that $\max\{\|x\|, \|y\|\} \le \|x+iy\|_{\mathbb{C}} = \|x-iy\|_{\mathbb{C}}$ and $\|x+i0\|_{\mathbb{C}} = \|x\|$ for all $x, y \in Y$). Then for each polynomial $P \in \mathcal{P}(^mY)$,

$$P^{\mathbb{C}}(x+iy) := \sum_{k=0}^{m} \binom{m}{k} i^{m-k} \check{P}(x^k, y^{m-k})$$

($\check{P}$ the symmetric linear mapping associated to $P$ (see [Di], Section 1.1)) which extends $P$ and fulfills $\|P^{\mathbb{C}}\| \le (2m)^m/m!\ \|P\|$. Conversely, for each $Q \in \mathcal{P}(^m(Y+iY))$ there is a unique $R \in \mathcal{P}(^m(Y+iY)_{\mathbb{R}})$ with

$$Q(x+iy) = R(x+iy) - iR(e^{\pi i/2m}(x+iy)); \tag{22}$$

in consequence $\|Q\| = \|R\|$.

We now prove (21). For each choice $P_1, \dots, P_M$ of polynomials in $\mathcal{P}(^mX(\mathbb{R})_n)$ we have

$$\begin{aligned}\left(\sum_{j=1}^{M} \|P_j\|^2\right)^{1/2} &\le \left(\sum_{j=1}^{M} \|P_j^{\mathbb{C}}\|^2\right)^{1/2} \\ &\le \mathbf{C}_2(\mathcal{P}(^mX_n)) \left(\int_0^1 \|\sum_{j=1}^{M} r_j(t) P_j^{\mathbb{C}}\|^2 dt\right)^{1/2}\end{aligned}$$

$$=\mathbf{C}_2(\mathcal{P}(^mX_n))\left(\int_0^1 \|\left(\sum_{j=1}^M r_j(t)P_j\right)^{\mathbb{C}}\|^2dt\right)^{1/2}$$

$$\leq\frac{(2m)^m}{m!}\mathbf{C}_2(\mathcal{P}(^mX_n))\left(\int_0^1 \|\sum_{j=1}^M r_j(t)P_j\|^2dt\right)^{1/2},$$

which obviously gives the first inequality in (21). For the second one we first check

$$\mathbf{C}_2(\mathcal{P}(^mX_n)) \leq \mathbf{C}_2(\mathcal{P}(^m(X_n)_{\mathbb{R}}));$$

indeed, for $Q_1,\ldots,Q_M \in \mathcal{P}(^mX_n)$ let $R_1,\ldots,R_M \in \mathcal{P}(^m(X_n)_{\mathbb{R}})$ be as in (22), hence

$$\left(\sum_{j=1}^M \|Q_j\|^2\right)^{1/2} = \left(\sum_{j=1}^M \|R_j\|^2\right)^{1/2}$$

$$\leq\mathbf{C}_2(\mathcal{P}(^m(X_n)_{\mathbb{R}}))\left(\int_0^1 \|\sum_{j=1}^M r_j(t)R_j\|^2dt\right)^{1/2}$$

$$=\mathbf{C}_2(\mathcal{P}(^m(X_n)_{\mathbb{R}}))\left(\int_0^1 \|\sum_{j=1}^M r_j(t)Q_j\|^2dt\right)^{1/2}.$$

Finally, we obtain from the symmetry of $X$ that $\|i\|\cdot\|i^{-1}\| \leq 4$, where the mapping

$$i\colon (X_n)_{\mathbb{R}} \to X(\mathbb{R})_{2n}$$

is defined by $i(x_1,\ldots,x_n) := (\operatorname{Re}x_1, \operatorname{Im}x_1,\ldots,\operatorname{Re}x_n,\operatorname{Im}x_n)$. Hence

$$\mathbf{C}_2(\mathcal{P}(^mX_n)) \leq \mathbf{C}_2(\mathcal{P}(^m(X_n)_{\mathbb{R}})) \leq 4^m\mathbf{C}_2(\mathcal{P}(^mX(\mathbb{R})_{2n})),$$

which completes the proof. ∎

Clearly, the complex versions of 4.1, 5.1 are now immediate, and the complex version of 6.1 follows the same way since by Remark 3.3 we have $\mathbf{M}_{(2)}(X(\mathbb{R})_n) \asymp \mathbf{M}_{(2)}(X(\mathbb{R})_{2n})$ (rewrite 6.1 in terms of concavity).

## 8. Estimates for concrete spaces

Let us now apply our results to get estimates for some special important sequence spaces. In view of the preceding section, now all considered Banach sequence spaces will be complex.

*Example 8.1:* We begin with $\ell_p$; it is well known that $\ell_p$ is $r$-convex iff $1 \le r \le p$, and $s$-concave iff $p \le s < \infty$. Therefore Theorem 5.1, Theorem 6.1 and (8) give, for each $m \in \mathbb{N}$,

$$\mathbf{C}_2(\mathcal{P}(^m\ell_p^n)) \asymp \begin{cases} \frac{(n^{1/2})^m}{\sqrt{\log(n+1)}} & \text{if } p = 1, \\ n^{m/2-1}n^{1/p} & \text{if } 1 < p \le 2, \\ (n^{1/2})^{m-1} & \text{if } 2 \le p \le \infty. \end{cases}$$

*Example 8.2:* Let $\varphi$ be a non-degenerate Orlicz function and $\ell_\varphi$ the Orlicz sequence space associated to it (see [14], Chapter 4, for definitions and properties). An Orlicz function $\varphi$ is said to satisfy the $\Delta_2$ condition if

$$\limsup_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \infty,$$

or equivalently, if there is a constant $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t \ge 0$. It is known (see [12], Corollary 13 and Corollary 15) that $\ell_\varphi$ is $s$-concave $(2 \le s < \infty)$ iff there is a constant $K > 0$ such that $\varphi(\lambda t) \ge K\lambda^s\varphi(t)$ for all $0 \le \lambda, t \le 1$. Moreover, it is $r$-convex $(1 < r \le 2)$ iff it satisfies the $\Delta_2$ condition and $\varphi(\lambda t) \le K\lambda^r\varphi(t)$ for all $0 \le \lambda, t \le 1$ and some $K > 0$. The fact that $\varphi$ satisfies the $\Delta_2$ condition guarantees that $\ell_\varphi$ has non-trivial concavity (see [12], Proposition 7). Finally note that an easy calculation shows that $\lambda_{\ell_\varphi}(n) = 1/\varphi^{-1}(1/n)$ for all $n \in \mathbb{N}$.

With all this we obtain by Theorem 5.1 and Theorem 6.1:

(i) Let $\varphi$ satisfy the $\Delta_2$ condition and be such that $\varphi(\lambda t) \le K\lambda^2\varphi(t)$ for all $0 \le \lambda, t \le 1$ and some $K > 0$; then

$$\mathbf{C}_2(\mathcal{P}(^m\ell_\varphi^n)) \asymp (n^{1/2})^{m-1}.$$

(ii) Let $\varphi$ be such that $\varphi(\lambda t) \ge K\lambda^2\varphi(t)$ for all $0 \le \lambda, t \le 1$ and some $K > 0$; then

$$\mathbf{C}_2(\mathcal{P}(^m\ell_\varphi^n)) \asymp \frac{n^{m/2-1}}{\varphi^{-1}(1/n)}.$$

*Example 8.3:* Let now $X = d(w,p)$ be a Lorentz space (see [14], Section 4.e, [15], Section 2.a for definitions and basic properties) and define $d^n(w,p) = X_n$. In [21] it can be found that $d(w,p)$ is always $r$-convex (and $\mathbf{M}_{(r)}(d(w,p)) = 1$) iff $1 \le r \le p$. In order to formulate concavity conditions, we say that $w$ is $\alpha$-regular $(0 < \alpha < \infty)$ if $w_n^\alpha \asymp \frac{1}{n}\sum_{i=1}^n w_i^\alpha$. Then [21], Theorem 2, shows that, for $p < s < \infty$, $d(w,p)$ is $s$-concave iff $w$ is $t/p$-regular, with $1/t = 1/p - 1/s$. It is non-trivially concave iff $w$ is 1-regular. In particular, this means that $d(w,p)$

is 2-convex for all $2 \leq p < \infty$; and that if $1 \leq p < 2$ and $nw_n^q \asymp \sum_{i=1}^n w_i^q$ with $q = 2/(2-p)$, it is 2-concave (and clearly of non-trivial convexity). Notice that if it is 2-concave, then $w$ is 1-regular.

This gives that for each $m \in \mathbb{N}$,

(i) if $2 \leq p < \infty$, then

$$\mathbf{C}_2(\mathcal{P}(^m d^n(w,p))) \asymp (n^{1/2})^{m-1};$$

(ii) if $1 \leq p < 2$ and $nw_n^q \asymp \sum_{i=1}^n w_i^q$ with $q = 2/(2-p)$, then

$$\mathbf{C}_2(\mathcal{P}(^m d^n(w,p))) \asymp n^{m/2-1}(\sum_{i=1}^n w_i)^{1/p} \asymp n^{m/2-1} n^{1/p} w_n^{1/p}.$$

*Example 8.4:* Let us consider now the case of Lorentz sequence spaces $\ell_{p,q}$ (see [2], [19]). It is known that $\ell_{p,q}$ is $r$-convex iff $r < p$, $r \leq q$ and it is $s$-concave if and only if $p < s$, $q \leq s$. With this,

$$\mathbf{C}_2(\mathcal{P}(^m \ell_{p,q}^n)) \asymp \begin{cases} n^{m/2-1} n^{1/p} & \text{if } 2 > p,\ 2 \geq q, \\ (n^{1/2})^{m-1} & \text{if } 2 < p,\ 2 \leq q. \end{cases}$$

## 9. Appendix

We give here a last result that, although it is not in the main trend of this paper, is based on some of the techniques that we have been using all through the paper. First of all notice that simply by joining Lemma 5.4 and Lemma 6.3 we have that if $X$ and $Y$ are two Banach sequence spaces, then for all $n, m$

$$(23) \qquad \|id\colon X_n \otimes_\varepsilon Y_m \to X_n(Y_m)\| \leq \min(\lambda_X(n),\ K_G \mathbf{M}_{(2)}(Y_m)\ m^{1/2}).$$

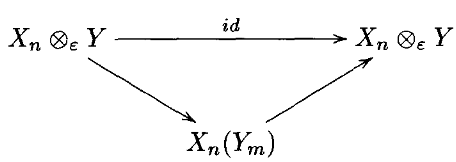
PROPOSITION 9.1: *Let $X$ be either 2-concave or 2-convex with non-trivial concavity and let $Y$ be either 2-concave or 2-convex with non-trivial concavity; then*

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \asymp \min(\sqrt{n}\mathbf{M}_{(2)}(Y_m),\ \sqrt{m}\mathbf{M}_{(2)}(X_n)),$$

*where this means that we can find upper and lower bounds with constants depending neither on $n$ nor on $m$.*

This is a proper improvement of a result on cotype 2 estimates for injective tensor products of $\ell_p^n$'s given in [1], Proposition in Section 5.

*Proof:* Let us begin by assuming that both spaces are 2-concave. We factorize,



Now, from this factorization and (23),

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \le K_G \mathbf{M}_{(2)}(Y_m)\ m^{1/2}\mathbf{C}_2(X_n(Y_m)).$$

Using again (5) we have some universal constant $K > 0$ such that $\mathbf{C}_2(X_n(Y_m)) \le K\mathbf{M}_{(2)}(X_n)\mathbf{M}_{(2)}(Y_m)$. Since $Y$ is 2-concave, the $\mathbf{M}_{(2)}(Y_m)$ are all bounded by $\mathbf{M}_{(2)}(Y)$, hence

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec\ m^{1/2}\mathbf{M}_{(2)}(X_n).$$

By the symmetry of $\varepsilon$, $\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec\ n^{1/2}\mathbf{M}_{(2)}(Y_m)$, which gives the desired upper estimate. To get the lower bound we have, from (11),

$$\sqrt{nm}\|\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m\| \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m) l(\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m).$$

We know that $\|\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m\| = \|\ell_2^n \to X_n\|\ \|\ell_2^m \to Y_m\|$. On the other hand, by Chevét's inequality (14),

$$l(\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m) \le c(l(\ell_2^n \to X_n)\|\ell_2^m \to Y_m\| + \|\ell_2^n \to X_n\| l(\ell_2^m \to Y_m)).$$

All this gives

$$(24) \qquad 1 \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\left(\frac{l(\ell_2^n \to X_n)}{\sqrt{nm}\ \|\ell_2^n \to X_n\|} + \frac{l(\ell_2^m \to Y_m)}{\sqrt{nm}\ \|\ell_2^m \to Y_m\|}\right).$$

Since $X$ has non-trivial concavity, by (2), $l(\ell_2^n \to X_n) \prec \lambda_X(n) \le \sqrt{n}\|\ell_2^n \to X_n\|$; and the same holds for $Y$. Sampling this in the last inequality and after the proper cancellations we obtain

$$1 \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\frac{1}{\min(\sqrt{m}, \sqrt{n})},$$

with which we have what we wanted. Assume now that both spaces $X$ and $Y$ are 2-convex and have non-trivial concavity. For the upper estimate we factorize as we did it before to get, applying Lemma 6.2, (5) and (23),

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \le \lambda_X(n)\mathbf{C}_2(X_n(Y_m)) \prec \sqrt{n}\mathbf{M}_{(2)}(Y_m).$$

But again we also have $\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec \sqrt{m}\mathbf{M}_{(2)}(X_n)$, which yields the upper bound. For the lower one we start, as before, from (11) and apply Chevét's inequality to arrive at (24). Then we apply that $\|\ell_2^n \to X_n\| \ge 1$ and, since $X$ is 2-convex and has non-trivial concavity, by (2) and Lemma 6.2, $l(\ell_2^n \to X_n) \asymp \lambda_X(n) \asymp \sqrt{n}\mathbf{M}_{(2)}(X_n)^{-1}$ (the same is true for $Y$) to get

$$\begin{aligned} 1 \prec &\mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\left(\frac{\sqrt{n}}{\sqrt{nm}\mathbf{M}_{(2)}(X_n)} + \frac{\sqrt{m}}{\sqrt{nm}\mathbf{M}_{(2)}(Y_m)}\right) \\ \prec &\mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\frac{1}{\min(\sqrt{m}\mathbf{M}_{(2)}(X_n), \sqrt{n}\mathbf{M}_{(2)}(Y_m))}. \end{aligned}$$

This proves the second case. The proof of the case when one is 2-concave and the other one is 2-convex with non-trivial concavity goes as the two previous ones, taking the appropriate estimate in each case. ∎

As a straightforward consequence we have the following characterization.

COROLLARY 9.2: *Let $X, Y$ be any two Banach sequence space; then $X, Y$ are both 2-concave if and only if*

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \asymp \min(\sqrt{n}, \sqrt{m}).$$

*Proof:* The 'only if' follows from Proposition 9.1. The 'if' implication yields from the fact that if we fix $m = 1$, then we have $X_n \otimes_\varepsilon Y_m = X_n$ and

$$\mathbf{C}_2(X_n) \asymp \min(\sqrt{n}, 1) = 1.$$

In other words, the $\mathbf{C}_2(X_n)$ are bounded and $X$ is not isometric to $\ell_\infty$. Therefore $X$ has cotype 2, hence is 2-concave. The same proof is valid for $Y$. ∎

It is well known that, if either $E$ or $F$ are finite dimensional, $\mathcal{L}(E;F) = E' \otimes_\varepsilon F$ holds isometrically. Therefore we obtain the following corollary.

COROLLARY 9.3: *Let $X$ be either 2-concave with non-trivial convexity or 2-convex with non-trivial concavity; then*

$$\mathbf{C}_2(\mathcal{L}(X_n)) \asymp \sqrt{n}.$$

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